Exercise 66

By the *end behavior* of a function we mean the behavior of its values as $x \to \infty$ and as $x \to -\infty$.

(a) Describe and compare the end behavior of the functions

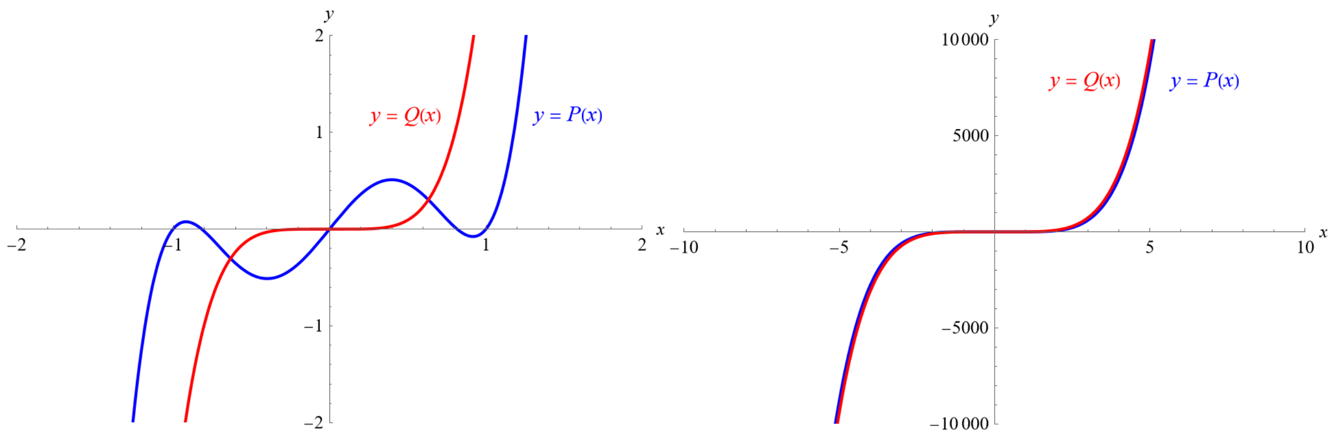
$$P(x) = 3x^5 - 5x^3 + 2x \qquad Q(x) = 3x^5$$

by graphing both functions in the viewing rectangles $[-2, 2]$ by $[-2, 2]$ and $[-10, 10]$ by $[-10{,}000, 10{,}000]$.

(b) Two functions are said to have the *same end behavior* if their ratio approaches 1 as $x \to \infty$. Show that P and Q have the same end behavior.

Solution

Below is a plot of both functions versus x on the different viewing windows.



Both the functions tend to $-\infty$ as $x \to -\infty$, and both the functions tend to ∞ as $x \to \infty$; in addition, as $|x|$ becomes large the functions appear to be parallel lines, which indicates they have the same end behavior. Calculate the limit of their ratio as $x \to \pm\infty$. In the second limit, make the substitution, $u = -x$, so that as $x \to -\infty$, $u \to \infty$.

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{3x^5 - 5x^3 + 2x}{3x^5} = \lim_{x\to\infty} \left(1 - \frac{5}{3x^2} + \frac{2}{3x^4}\right) = 1 - 0 + 0 = 1$$

$$\begin{aligned}
\lim_{x\to-\infty} \frac{P(x)}{Q(x)} &= \lim_{u\to\infty} \frac{3(-u)^5 - 5(-u)^3 + 2(-u)}{3(-u)^5} \\
&= \lim_{u\to\infty} \frac{-3u^5 + 5u^3 - 2u}{-3u^5} \\
&= \lim_{u\to\infty} \left(1 - \frac{5}{3u^2} + \frac{2}{3u^4}\right) \\
&= 1 - 0 + 0 \\
&= 1
\end{aligned}$$